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Extensional singularity dimensions for strange attractors

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A new set of dimension-like invariants is obtained, which characterize aspects of the attractor not usually examined. This study also demonstrates a new method for determining Lyapunov exponents and introduces "tailored attractors" for which the Lyapunov exponents and fractal dimension may be precisely set.

In this paper we examine some new invariants that characterize a strange attractor. In a recent study on the calculation of Lyapunov exponents from time series data [1], it was shown that one could examine the singularity spectrum of local neighborhoods of data vectors. A particular neighborhood contains all data vectors from the entire data set within a certain radius of a given vector. However, due to the fractal nature of the attractor these neighborhoods are filled in a highly nonuniform manner. The singularity analysis identifies directions in the data vector space in which the data set is very thin or compressed relative to the radius of the neighborhood. For d dimensional vectors, d singular values or "thicknesses" are determined which characterize the data set in that neighborhood. By examining the behavior of these singular thickness values as the size of the neighborhood is decreased, we can obtain a series of dimension-like invariants which characterize the attractor, the first few of which are approximately equal to the usual fractal dimension. The remaining dimensions may have quite different values which are smaller than the first, and pertain to aspects of the attractor which are not usually examined. These are related to "extra directions" in the phase space for which the attractor has no Cantor set or continuous structure. However, the data set may still exhibit weak extensions into these directions due to a "fractal texture" or bumpiness of the attractor. The extensional dimensions show in a specific way how this fractal texture tends to get squeezed out or lost as the attractor is examined at smaller and smaller scales. The weak extensions of the attractor are essential for determining important information about these extra directions of the phase space such as the associated Lyapunov exponents.

In addition, this paper describes a new method for determining Lyapunov exponents, makes some important comments regarding the concept of embedding dimension, and utilizes a new method for creating "tailored attractors" for which the Lyapunov exponents may be precisely set.

Note that the extensional dimensions are quite different from the set of generalized dimensions which occur as a result of the multifractal nature of the attractor [2] and the related function $f(\alpha)$ [3]. They are also quite different from the partial dimensions of the attractor [4]. In typical cases one will find that there will be one or more partial dimensions with a value of unity, a single one with a value between zero and one, and all the rest will have a value of zero. As might be expected from their name the partial dimensions are usually expected to add up to the full fractal dimension of the attractor, and thus for the typical cases they contain no new information. The directions for which the partial dimension is zero are the extra directions mentioned pre-

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viously for which the most interesting values are obtained for the extensional dimensions.

The tailored attractors are generated using a simple nonlinear system such as the logistic map to drive a linear subsystem in a manner similar to that used by Pecora and Carroll [5]. If y_1 represents an output from the driver system then the linear subsystem consists of the variables z_i which satisfy the equations

$$z'_i = c_i(z_i + y_1) , \tag{1}$$

where z'_i is the forward iterate of z_i and the c_i are parameters chosen to tailor the behavior of the combined system. The Lyapunov exponents of the full system will include the exponents of the driver system (unchanged), plus the exponents from the linear subsystem whose values are given by $\ln|c_i|$. This must be true since any perturbation of the ith subsystem variable will simply decay by the factor c_i each iteration, while having no effect on the other variables. For the form given above all of the subsystem exponents must be negative for stable operation of the map. In order to add additional positive exponents, eq. (1) would need to be converted to another form such as a tent or sawtooth map, which would prevent z_i from becoming arbitrarily large. Such a map could also be used as the driver giving precise control over all of the Lyapunov exponents. Note that similar results were obtained by Badii et al. [6], who studied the effect of filtering on data from dynamical systems.

If it is desired to have an output in the form of a scalar time series x_n, as is used in this paper, it is necessary to make some linear combination of the driver and subsystem variables. We use $x = y_1 + \sum z_i$. If the sum of the driver's exponents is negative, and the subsystem exponents are all more negative than those of the driver then the fractal dimension of the full space attractor will remain essentially unchanged from that of the driver by itself. This follows from the Kaplan–Yorke formula [7] for the Lyapunov fractal dimension. Let the Lyapunov exponents, λ_i, be ordered so that $\lambda_i > \lambda_j$ for $i<j$, and let k be the maximum number of the Lyapunov exponents which can be added together before the sum becomes negative. The Lyapunov dimension f_L is defined by

$$f_L = k + \frac{\sum_{m=1}^{k} \lambda_m}{-\lambda_{k+1}} . \tag{2}$$

If, however we choose one or more of the new exponents to be less negative, it is possible to change the dimension of the attractor, even though the essential dynamics (as determined by the driver) remain unchanged. This is illustrated in fig. 1, where the fractal dimension of the Hénon attractor [8] has been increased to 2.14 from its initial value of 1.25. The Hénon map is given by

$$y'_1 = 1 + y_2 - a y_1^2 , \tag{3}$$

$$y'_2 = b y_1 , \tag{4}$$

where we used the commonly chosen values for the parameters: $a = 1.4$ and $b = 0.3$. This system yields exponents of 0.408 and -1.58. To this was added a two-dimensional subsystem with exponents -0.3 and -0.8. In the case of a driver for which the sum of the Lyapunov exponents is positive, the dimension of the attractor is equal to the phase space dimension. When a subsystem is added, the dimension of the attractor can be increased, and fractal structure begins to appear if the sum becomes negative. This is illustrated for the case of the logistic map in fig. 2, where the dimension is increased from the usual dimension of 1 to 1.71. The logistic map is given by

$$y'_1 = a y_1 (1 - y_1) . \tag{5}$$

We chose the value $a = 3.7$ for which the positive exponent is approximately 0.356. To this we added a

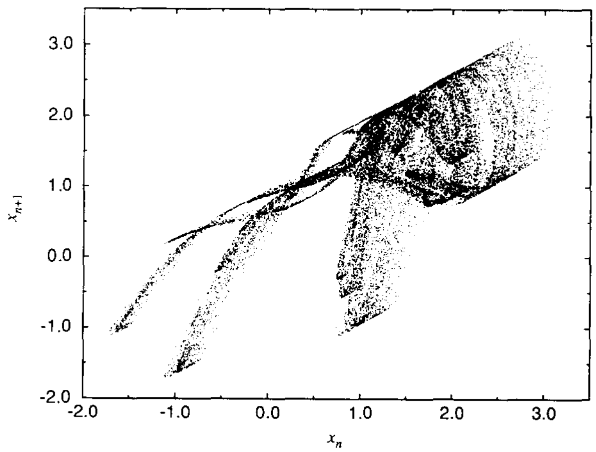

Fig. 1. Return map for the Hénon map driving a linear subsystem. The dimension of the attractor has been increased from 1.25 to 2.14 by the additional Lyapunov exponents.

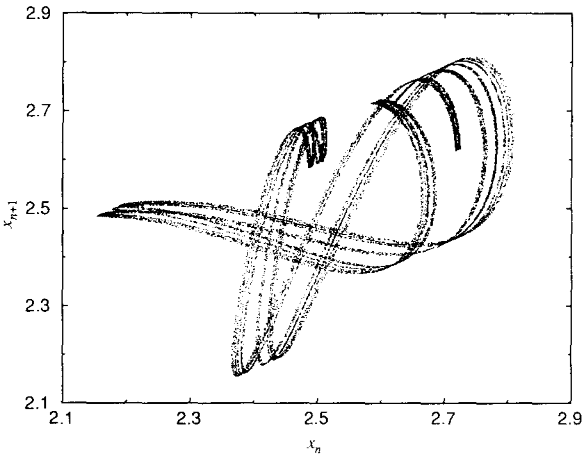

Fig. 2. Return map for the logistic map driving a linear subsystem. The attractor dimension is increased from the usual value of 1.00 to 1.71 by the action of the subsystem and, for the first time, we can observe fractal structure for this map.

three-dimensional subsystem with exponents -0.5, -0.7, and -1.5.

In order to analyze the singularity of the local neighborhoods, two methods were devised, each of which was found to give similar results. In the first method, we try to find a (hyper)surface of dimension one less than that of the data vectors that has a minimum r.m.s. displacement from the vectors. This displacement is then a measure of the data set singularity. One then proceeds by projecting out this most singular direction and repeating the calculation with one less dimension obtaining a second singular thickness. Continuing in this fashion, one obtains d thickness values from the d-dimensional vectors that one started with. Note that the surfaces do not need to be flat, one can do considerably better by defining these surfaces by a general polynomial of some order n_{tay} which gives the displacement of the optimal curved surface along the singular direction relative to the flat surface. The number of neighbors n_{nb} required to obtain all of the coefficients which define the surface increases quite rapidly with n_{tay}. (Note: it is generally recommended to use about twice the minimum number of neighbors.) The error in determining the thickness of the data set should be proportional to $r^{n_{tay}+1}$, where r is the r.m.s. displacement of the data points from the original point whose neighbors we are studying. As we will show, the singular thickness values often decrease more rapidly than r as we look at smaller and smaller neighborhoods. Clearly then we need to choose n_{tay} large enough so that the error decreases more rapidly than the thickness which we are measuring. When the size of a neighborhood is decreased, n_{tay} determines the maximum rate of singular thickness decrease which can be followed.

In the second method, we use a variant of singular value decomposition (SVD) [9]. For a first order calculation ($n_{tay}=1$), we start by making the matrix A out of all of the displacement vectors of the neighbors from the original point so that $A_{i,j}$ is the jth component of the ith vector. Using SVD, this matrix is factored in the form $A=UWV^{T}$, where U is an column orthogonal matrix, W is a diagonal matrix which contains the d singular values on its diagonal, and V is a square orthogonal matrix of dimension d. The d singular values, when divided by $\sqrt{n_{nb}}$, yield the same r.m.s. thickness values which we could have calculated by the first method. To extend this to higher order, the column vectors of A are first orthogonalized with respect to nonlinear terms through a certain order. The details of this analysis will be presented in a longer paper, now in preparation.

We will now study the behavior of these singularities as we decrease the size of the neighborhoods. The number of neighbors n_{nb} is fixed, but we increase the total number of points n_{pts} in the data set from 10^3 to 10^8. The logarithms of the singularities are averaged over a number of neighborhoods (usually 100) for a given data set. Only the points in these neighborhoods are saved, making it much easier to work with extremely large data sets. In fig. 3, we show singular thickness values S_i obtained from the logistic map (5) and subsystem, set as previously for Lyapunov exponents of 0.356, -0.5, -0.7, and -1.5. Using the time lag method [10,11,1], data vectors were obtained by taking sets or four consecutive data points from the scalar time series x_n. The slopes of the curves give the "extensional singularity dimensions", f_i, for this data set. The singularity analysis was carried out using the second method with $n_{tay}=3$ and $n_{nb}=68$. The first two curves have nearly the same slope, $f_1=1.66$ and $f_2=1.72$, and are approximately equal to the Lyapunov dimension from eq. (2) ($f_L=1.71$). It is generally found that this will be true for the first $k+1$ dimensions. The remaining

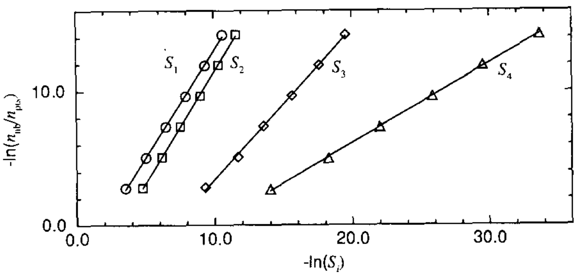

Fig. 3. The decay of singular thickness values with decreasing neighborhood size for data from the logistic map and 3-D linear subsystem. Note that the slopes of the first two curves are approximately equal to the ordinary fractal dimension of 1.71.

dimensions are significantly smaller however: $f_3 = 1.16$ and $f_4 = 0.59$. This indicates that the weak extensions of the attractor in these directions are getting squeezed out as the attractor is examined on smaller and smaller scales, and provides us with a quantitative measure of this behavior.

The values of these dimensions can be related to the Lyapunov exponents by analyzing the deformation of a neighborhood as it is iterated forward in time. Assume that we can identify the directions in the neighborhood space which are associated with the various Lyapunov exponents of the system. The j positive values are associated with stretching of the space, so that any structures in the attractor will extend to the edges of the neighborhood (and beyond) along these directions. A minimum of $k+1$ directions are required for dissipation so that the sum of the exponents through λ_{k+1} is negative. Examining a cross section of the neighborhood space for directions $j+1$ through $k+1$ will reveal a fractal structure, which will be compressed as the neighborhood is iterated forward in time. The volume of this subspace will decrease at the rate governed by the sum of λ_{j+1} through λ_{k+1}. Assuming that this subspace is in some (statistical) sense isotropic, this is equivalent to decreasing the radius of our neighborhood at the rate given by the average of the exponents $j+1$ through $k+1$. Meanwhile any displacements in directions beyond $k+1$ are decreasing more rapidly at the rates given by their own exponents. From these relative scaling rates we obtain an expression for the ith extensional singularity dimension,

$$f_i = \frac{f_L \sum_{m=j+1}^{k+1} \lambda_m}{\lambda_i (k+1-j)} , \qquad (6)$$

for $i > k+1$. In cases where $j = k$ such as the one previously discussed, the formula reduces to $f_i = f_L \lambda_{k+1} / \lambda_i$. This simpler form also does not depend on our assumption of an isotropic subspace since the subspace is now one-dimensional. Thus we obtain for the previous case $f_3 = 1.223$ and $f_4 = 0.571$ which are in reasonable agreement with the numerical values above (numerical error may be about 10%).

In fig. 4 we show a case where $j \neq k$. This shows singularity data from the Hénon map [5] and subsystem, set as previously for Lyapunov exponents of 0.408, -0.3, -0.8, and -1.58. From the slopes we estimate $f_1 = 2.05$, $f_2 = 2.15$, $f_3 = 1.74$, and $f_4 = 0.75$. The first three are in fairly good agreement with the Lyapunov dimension: $f_L = 2.14$, and the fourth is close to the expected value 0.743 obtained from eq. (6).

In order to insure that the concepts developed here remain valid outside the realm of tailored attractors, some studies have been conducted using ordinary (nontailored) attractors. Here we present some results from the four-dimensional map

$$w' = (1 + x - aw^2 + cy)/(1 + dw^2) , \qquad (7)$$

$$x' = bw , \qquad (8)$$

$$y' = (1 + z - ey^2 + gw)/(1 + hy^2) , \qquad (9)$$

$$z' = fy , \qquad (10)$$

where a through h are parameters. For $d = h = 0$, this

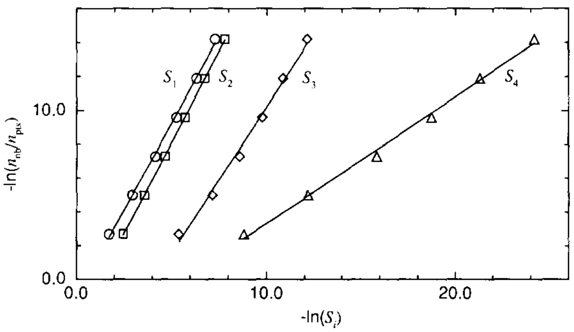

Fig. 4. This figure shows the decay of singular thickness values with decreasing neighborhood size for data from the Hénon map and 2-D linear subsystem. Note that the slopes of the first three curves are approximately equal to the ordinary fractal dimension of 2.14.

is a coupled dual Hénon map. For parameter values $a=2.2$, $b=0.4$, $c=0.2$, $d=1.0$, $e=0.9$, $f=0.25$, $g=0.2$, and $h=1.0$, the mapping has an attractor with Lyapunov exponents 0.332, −0.586, −1.320, −1.665 (accuracy ±0.005). From the slopes of the singularity plots (not shown) following values were obtained for the extensional singularity dimensions: $f_1=1.41$, $f_2=1.49$, $f_3=0.68$, and $f_4=0.53$. The first two are in fairly good agreement with the Lyapunov dimension: $f_L=1.57$. The third and fourth are close to the expected values 0.70 and 0.55 obtained from eq. (6).

In addition to their importance in characterizing the structure of a chaotic attractor, from eq. (6) we see that measurement of the extensional singularity dimensions can be used as a method for obtaining approximate values for Lyapunov exponents beyond λ_{k+1}, provided we have values for λ_{j+1} through λ_{k+1} and f_L. Using a local dimension $d=k+1$ for observing the attractor, one can (for clean data) obtain approximate values for first $k+1$ exponents by the conventional method [1], since, for sufficiently small neighborhoods, there will be relatively little extension in higher dimensions due to the singularity effects we have been describing. Note that the neighbors should first be located using a global dimension [1] D_G which satisfies $D_G>2f$ where f is the usual box-counting dimension in order to reduce the possibility of self intersection of the attractor [1,12]. Increasing d in unit steps one might then proceed to try and calculate succeeding extensional singularity dimensions and use these in eq. (6) to estimate the corresponding Lyapunov exponent. The result may then be compared with an attempt by the conventional method to obtain the same exponent. Reasonable agreement would lend considerable credibility to the calculated value, since the two methods are quite different. In principle, this procedure can be continued until d reaches the dimension of the original phase space D. Thus for an infinite dimensional system there is no well-defined limit to d. In effect, this means that there is no minimum embedding dimension for continuum systems; some information is always lost by choosing a finite d. In practice however, the maximum useful value for d may often be determined by the noise of the system; if singularity measurements beyond a certain dimension decay too rapidly below the noise floor [#1], then those additional dimensions can be expected to yield little useful dynamical information.

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[#1] Some previous results on the noise floor and the use of SVD can be found in ref. [13].

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